J. Fluid Mech. (1978), *vol.* 86, *part* 4, *pp.* 745–759
Printed in Great Britain

Calculation of axisymmetric jets and wakes with a three-equation model of turbulence

By SEDAT BIRINGEN
Von Kármán Institute, Rhode-St-Genèse, Belgium†

(Received 31 January 1977 and in revised form 12 December 1977)

The concept of diffusion by bulk convection formulated by Bradshaw is applied to the transport equations for the turbulent kinetic energy, turbulent shear stress and an integral length scale. The resulting set of hyperbolic partial differential equations is solved by an explicit finite-difference scheme for the cases of incompressible axisymmetric wakes and jets in a coflowing air stream. It is found that the profiles of mean velocity and shear stress are almost insensitive to the empirical input whereas the profiles of kinetic energy are very sensitive.

1. Introduction

One major problem in turbulence-field modelling techniques is the choice of necessary and adequate equations. Considering the turbulent kinetic energy equation to be the only equation sufficiently documented by experiment, Bradshaw, Ferriss & Atwell (1967) have transformed it into a rate equation for the shear stress. The basic assumption in this formulation is that profiles of all turbulence quantities at a given streamwise location are related to the shear-stress profile by empirical functions. The outstanding feature is the modelling of the diffusion term by bulk convection, which makes the resulting set of equations of hyperbolic type. The original method was applied to three-dimensional boundary layers by Nash (1969, 1972). An adapted version using the interaction hypothesis to deal with shear layers in which the shear-stress changes sign was used by Bradshaw, Dean & McEligott (1972) to calculate symmetrical duct flows. Further, the same hypothesis was used by Morel & Torda (1974) to calculate two-dimensional free shear flows, but with the inclusion of an empirical rate equation for the length scale. The interaction hypothesis mainly considers the flow as two separate interacting shear layers which are assumed to interact only through the mean velocity profile. As pointed out by Morel & Torda (1974), although this is a plausible approach for two-dimensional flows its applicability to the axisymmetric case is questionable owing to the necessity of considering an infinite number of interactions.

The simultaneous use of the equations for the turbulent kinetic energy and the shear stress constitutes the approach which is generally referred to as the multi-equation model of turbulence. Turbulence models in this class have been developed by Daly & Harlow (1970), Donaldson (1971), Rotta (1971, 1975), Hanjalić & Launder (1972), Lumley & Khajeh-Nouri (1974) and Launder, Reece & Rodi (1975). All have the

† Present address: Mechanical Engineering Department, Boğaziçi University, P.K. 2, Bebek, Istanbul, Turkey.

common feature that diffusion of kinetic energy or Reynolds stress is represented as a gradient transport phenomenon. Donaldson (1971) has assumed the dissipation length scale of turbulence to be a constant, whereas Rotta (1971) has used a rate equation to represent it. Daly & Harlow (1970), Hanjalić & Launder (1972), Lumley & Khajeh-Nouri (1974) and Launder *et al.* (1975) have expressed the turbulent dissipation term explicitly in terms of its own rate equation. All of these methods use the basic formulation of Rotta (1951) to represent the pressure/velocity-gradient correlations appearing in the shear-stress transport equation. This term represents a major difficulty in the multi-equation models of turbulence owing to the impracticability of its measurement and therefore its quantitative evaluation.

The present work is an attempt to apply the concept of diffusion by bulk convection to the rate equations of the turbulent shear stress, turbulent kinetic energy and an integral length scale. The resulting set of hyperbolic partial differential equations is solved by the Lax–Wendroff explicit finite-difference scheme for the cases of incompressible axisymmetric wakes and jets in a coflowing air stream. Physically, the coflowing jet can be considered as a relatively complex turbulent flow which cannot be put in self-preserving form. This, together with the absence of solid boundaries, makes these flows attractive cases for testing models of turbulence.

2. The governing equations and closure assumptions

The governing equations are written with respect to mutually perpendicular co-ordinates x, y and z such that for the axisymmetric case $y \equiv r\theta$. The mean velocity components are (U, V, W) and the fluctuating velocity components (u, v, w). An overbar denotes a time average, a repeated index indicates summation, subscript 1 refers to free-stream conditions whereas a subscript 0 refers to the centre-line. If the usual boundary-layer approximations are made and account is taken of the fact that at high Reynolds numbers the turbulent stresses are much larger than the molecular stresses, the equations of continuity, momentum, turbulent kinetic energy and turbulent shear stress take the following forms (see, for example, Rotta 1971):

$$\partial(Uy^r)/\partial x + \partial(Vy^r)/\partial y = 0, \tag{1}$$

$$\left.\begin{aligned} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{y^r}\frac{\partial}{\partial y}(y^r\overline{uv}) + \frac{\partial}{\partial x}(\overline{u^2} - \overline{v^2}) + \frac{1}{\rho}\frac{dP_1}{dx} = 0, \\ P_1 + \tfrac{1}{2}\rho U_1^2 = \text{constant}, \end{aligned}\right\} \tag{2}$$

$$\underbrace{U\frac{\partial k}{\partial x} + V\frac{\partial k}{\partial y}}_{\text{convection}} = \underbrace{-\overline{uv}\frac{\partial U}{\partial y}}_{\text{production}} \underbrace{-\frac{1}{y^r}\frac{\partial}{\partial y}\overline{y^r v\left(k + \frac{p}{\rho}\right)}}_{\text{diffusion}} \underbrace{-\epsilon}_{\text{dissipation}}, \tag{3}$$

where $k \equiv \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2})$,

$$\underbrace{U\frac{\partial \overline{uv}}{\partial x} + V\frac{\partial \overline{uv}}{\partial y}}_{\text{convection}} = \underbrace{-\overline{v^2}\frac{\partial U}{\partial y}}_{\text{production}} + \underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}}_{\text{redistribution}} \underbrace{-\frac{\partial}{\partial y}\overline{\left(v^2 + \frac{p}{\rho}\right)u}}_{\text{diffusion}} \underbrace{-2\nu\overline{\left(\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right)}}_{\text{dissipation}}. \tag{4}$$

In the momentum equation the normal-stress term is smaller than the others and hence will be neglected, and for the cases under investigation $dP_1/dx = 0$. It should further be noted that the above form of the shear-stress rate equation implies that the term pertaining to the effect of the 'redistributive' part of the shear-stress diffusion is neglected because it is smaller than the other terms. In the above equations $r = 1$ for axisymmetric flows and $r = 0$ for two-dimensional flows.

To close the set of equations (1)–(4), the diffusion and dissipation terms in (3) and the production, diffusion and redistribution terms in (4) have to be modelled.

The dissipation term is generally modelled by using the Kolmogorov hypothesis of locally isotropic turbulence at high Reynolds numbers. This, as well as the fact that in isotropic turbulence cross-correlations between velocity components must be zero, leads to

$$\epsilon = \nu \overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} + \nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} = \nu \overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} = C_1 \frac{k^{\frac{3}{2}}}{L}, \quad 2\nu \overline{\left(\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right)} = 0, \tag{5}$$

where C_1 is a constant and L is assumed to be proportional to an integral length scale representative of the large-scale motion.

The diffusion terms in (3) and (4) contain both third-order velocity correlations and pressure–velocity correlations. The latter are neglected because the few measurements of transport-equation terms suggest that the correlations are small near a free-stream edge, which in turn implies that they are not much larger in the interior of the flow field (Bradshaw *et al.* 1967; Hanjalić & Launder 1972), so that

$$\overline{up}/\rho = \overline{vp}/\rho = 0. \tag{6}$$

The diffusion of velocity fluctuations can be accepted as consisting of two parts, namely a gradient diffusion and a bulk convection. If ϕ is the diffused velocity correlation then

$$\overline{\phi u_i} = C u_T L_T \, \partial\overline{\phi}/\partial x_i + C_2 (V_\phi)_i \overline{\phi}, \tag{7}$$

where C and C_2 are constants, u_T is a characteristic velocity of the turbulence and $(V_\phi)_i$ is a turbulent transport velocity which is characteristic of the large-scale motion. The relative amounts of transport by gradient diffusion and bulk convection depend on the quantity transported (Townsend 1956, p. 110). Following Bradshaw *et al.* (1967), the assumption that $C = 0$ in (7) leads to the following expressions for the diffusion terms in (3) and (4):

$$\overline{vk} = C_2 V_k k, \quad (\overline{uv^2}) = C_2' V_j \overline{uv}. \tag{8}$$

In (8) the subscripts k and j refer to the quantities k and $\overline{uv}$ respectively; C_2 and C_2' are constants to be determined.

The general expression for the redistribution term in (4) was obtained by Chou (1945) by taking the divergence of the equation for the velocity fluctuations. This equation shows that the pressure–velocity correlations originate from interactions of the velocity fluctuations among themselves and the interaction of the mean velocity gradients with the velocity fluctuations. For flows with a preferred direction of the mean velocity, Rotta (1971) has proposed the following expression:

$$\overline{\frac{p}{\rho}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)} = -C_1 C_3 \frac{k^{\frac{1}{2}}}{L} \overline{uv} + C_4 k \frac{\partial U}{\partial y}, \tag{9}$$

in which C_3 and C_4 are constants to be determined experimentally. It should also be noted that in thin shear flows the model proposed by Hanjalić & Launder (1972) reduces essentially to the form given by (9).

Following Hanjalić & Launder (1972), the normal-stress terms will be assumed to be proportional to k. This gives

$$\overline{v^2}\,\partial U/\partial y = C_5 k\,\partial U/\partial y, \tag{10}$$

where C_5 is a constant to be determined experimentally. Finally, substitution of (5), (6), (8), (9) and (10) into (3) and (4) yields the approximate forms of the governing equations.

Integration of the rate equation for the two-point velocity correlation $\overline{u_{i,A}u_{j,B}}$ gives transport equations for an integral length scale times $\overline{u_i u_j}$. Rotta (1951) has derived an equation for an integral length scale defined by

$$2kL = \frac{3}{8}\int_{-\infty}^{+\infty} R_{ii}(y)\,dy, \tag{11}$$

where R_{ii} is the two-point correlation function and $2k = R_{ii}(0)$. The factor $\frac{3}{8}$ makes L identical with the lateral integral scale in an isotropic turbulence field. If only the first term is retained in the series expansion for the production term, if the dissipation term is assumed to be proportional to the dissipation term in the kinetic energy transport equation and if the diffusion is modelled by bulk convection, Rotta's equation yields

$$\underbrace{U\frac{\partial \overline{kL}}{\partial x} + V\frac{\partial \overline{kL}}{\partial y}}_{\text{convection}} = \underbrace{C_6\overline{uv}L\frac{\partial U}{\partial y}}_{\text{production}} - \underbrace{C_2''\frac{1}{y^r}\frac{\partial}{\partial y}y^r kLV_L}_{\text{diffusion}} - \underbrace{C_1C_8k^{\frac{3}{2}}}_{\text{dissipation}}. \tag{12}$$

Equation (12) can be further simplified by subtracting $L(U\,\partial k/\partial x + V\,\partial k/\partial y)$ from both sides and dividing by k. Neglecting the effects of the mean strain rate in comparison with those of the fluctuating strain rate and assuming $V_L = V_k$ and $C_2'' = C_2$ (Bradshaw 1973) gives the following rate equation for L:

$$U\frac{\partial L}{\partial x} + V\frac{\partial L}{\partial y} = -C_2V_k\frac{\partial L}{\partial y} - C_1(C_8 - 1)\,k^{\frac{1}{2}}, \tag{13}$$

where C_8 is a constant to be determined.

The primary objective in assigning values to the constants appearing in the closure assumptions is that they should be applicable to a large number of flow cases. Therefore simple types of flows are considered in order to evaluate some of these constants. Having noted that Uberoi's (1957) measurements of the decay of an isotropic field in the absence of mean shear indicate that $2{\cdot}6 < C_3 < 3{\cdot}0$, Rotta (1975) recommends $C_3 = 2{\cdot}8$. The same value is used by Hanjalić & Launder (1972). Considering a homogeneous turbulent field in which the only non-zero velocity component U increases linearly with y, the approximate forms of the equations for the kinetic energy and shear stress give $C_5 - C_4 = C_3(\overline{uv}/k)^2$. Hanjalić & Launder (1972) have estimated $(\overline{uv}/k)^2 = 0{\cdot}1$ from the measurements of Champagne, Harris & Corrsin (1970). For the dissipation constant Rotta (1975) has estimated $C_1 = 0{\cdot}165$. Further, from his inductive treatment of the length-scale equation, a value of $C_8 = 0{\cdot}8$ is obtained.

The bulk velocities for k and $\overline{uv}$ can be estimated by considering V_k and V_j to be simple functions of x and y (Bradshaw *et al.* 1967) such that

$$V_k = C_2\,|\overline{uv_m}|^{\frac{1}{2}}f_1(y), \quad V_j = C_2'\,|\overline{uv_m}|^{\frac{1}{2}}f_2(y). \tag{14}$$

In these expressions $|\overline{uv_m}|^{\frac{1}{2}}$ is chosen to be a velocity scale of the large eddies. The symmetry conditions for the distributions of the second and third-order correlations require both $f_1(y)$ and $f_2(y)$ to be antisymmetric.

3. The calculation procedure

The approximate forms of the governing equations together with the length-scale equation constitute the following system:

$$\frac{\partial \mathbf{f}}{\partial x}+\mathbf{A}(\mathbf{f},x,y)\frac{\partial \mathbf{f}}{\partial y}+\mathbf{B}(\mathbf{f},x,y)=0. \tag{15}$$

Here **f** and **B** are four-component vectors and **A** is a 4×4 matrix:

$$\mathbf{f}=\begin{bmatrix} U \\ k \\ \overline{uv} \\ L \end{bmatrix},\quad \mathbf{A}=\frac{1}{U}\begin{bmatrix} V & 0 & 1 & 0 \\ \overline{uv} & V+C_2V_k & 0 & 0 \\ ak & 0 & V+C_2'V_j & 0 \\ 0 & 0 & 0 & V+C_2V_k \end{bmatrix},$$

$$\mathbf{B}=\frac{1}{U}\begin{bmatrix} \overline{uv}/y^r \\ C_2k\,\partial V_k/\partial y+(C_2V_kk/y)\,r+C_1k^{\frac{3}{2}}/L \\ C_2'\overline{uv}\,\partial V_j/\partial y+C_1C_3k^{\frac{1}{2}}\,\overline{uv}/L \\ C_1(C_8-1)\,k^{\frac{1}{2}} \end{bmatrix},$$

where $a\equiv C_5-C_4$. The cross-stream mean velocity V appears only in the coefficient matrix **A**, and hence can be calculated separately from the equation along the vertical characteristic. This equation is obtained by substituting the equation of continuity into the x-momentum equation and reads

$$\frac{1}{y^r}\frac{d(y^rV)}{dy}-\frac{V}{U}\frac{dU}{dy}-\frac{1}{Uy^r}\frac{d(y^r\overline{uv})}{dy}=0. \tag{16}$$

In (15) and (16), $r=1$ for axisymmetric flows and $r=0$ for two-dimensional flows.

The characteristic equation of the matrix **A** is

$$\det(\lambda\mathbf{I}-\mathbf{A})=0, \tag{17}$$

where **I** is the identity matrix. The eigenvalues of **A** are found from (17) and read

$$\left.\begin{aligned} \lambda_{1,2}&=(V+C_2V_k)/U \quad \text{(double characteristic)},\\ \lambda_{3,4}&=\{2V+C_2'V_j\pm[(C_2V_j)^2+4ak]^{\frac{1}{2}}\}/2U. \end{aligned}\right\} \tag{18}$$

Since the quantity $4ak$ is always positive, all the eigenvalues are real and correspond to real characteristic directions. Under this condition the system of equations (15) is hyperbolic.

The boundary conditions imposed at the inner and outer boundaries are as follows: at the inner boundary (the line of symmetry), $\overline{uv}$ and cross-stream derivatives of U, k and L are set to zero; at the outer edge, the free-stream conditions are assumed to be $U=U_1$, $k=0$ and $\overline{uv}=0$ and the cross-stream derivative of L is set to zero. Following Bradshaw *et al.* (1967), the outer edge of the flow is taken as the point where the difference between two successive values of the mean velocity is less than a certain specified amount.

In the axisymmetric case there are two terms in the vector **B** which will assume the indeterminate form 0/0 at the inner boundary, where $y=0$. Limiting forms for these terms can be evaluated by series expansion such that

$$\lim_{y\to 0}\frac{\overline{uv}(y)}{y}=\frac{\partial\overline{uv}}{\partial y}(0),\quad \lim_{y\to 0}\frac{V_k(y)\,k(y)}{y}=k(0)\frac{\partial V_k}{\partial y}(0). \tag{19}$$

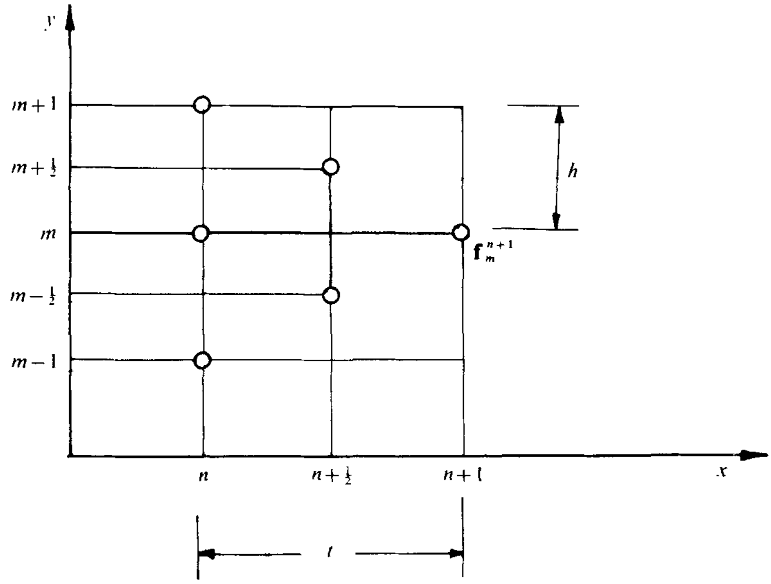


FIGURE 1. The Lax–Wendroff computational molecule.

It has already been stated in § 2 that V_k is an antisymmetric function of y. This adds the constraint that $V_k(0) = 0$.

In order to integrate the set of governing equations (15) the explicit two-step Lax–Wendroff method was used. This method consists mainly of using Lax's method to calculate the provisional values of the unknown vector $\mathbf{f}$ at the intermediate marching levels. These values are then used to calculate the primary points by employing a midpoint leapfrog calculation (Richtmyer & Morton 1967, p. 303). A typical computational molecule is shown in figure 1; using the same notation as in figure 1, the two steps of this method can be written as

$$\mathbf{f}_{m+\frac{1}{2}}^{n+\frac{1}{2}} = \tfrac{1}{2}(\mathbf{f}_{m+1}^{n}+\mathbf{f}_{m}^{n}) + \tfrac{1}{2}t(\partial\mathbf{f}/\partial x)_{m+\frac{1}{2}}^{n}, \tag{20}$$

$$\mathbf{f}_{m}^{n+1} = \mathbf{f}_{m}^{n} + t(\partial\mathbf{f}/\partial x)_{m}^{n+\frac{1}{2}}. \tag{21}$$

The original Lax–Wendroff scheme employs only second-order-accurate finite-difference operators. Nash (1972), however, reports the occurrence of instabilities in his calculation of a three-dimensional boundary layer using Bradshaw's turbulence model and the Lax–Wendroff scheme. Stability was restored by using fourth-order-accurate central differences in the cross-stream direction. He further reports that maximum precision was obtained when fourth-order differences were used only in the first step. Maximum stability corresponded to the case when the values of $\mathbf{f}$ at the advanced levels were calculated by using only provisional values from the intermediate marching levels. Making use of (20) and (21) and employing fourth-order central differences in the first step, (15) can be put into the following difference form:

$$\mathbf{f}_{m+\frac{1}{2}}^{n+\frac{1}{2}} = \tfrac{1}{2}(\mathbf{f}_{m+1}^{n}+\mathbf{f}_{m}^{n}) - \frac{t}{2h}\mathbf{A}_{m+\frac{1}{2}}^{n}(\mathbf{f}_{m+1}^{n}-\mathbf{f}_{m}^{n}) + \frac{t}{2h}\frac{\Omega}{24}\mathbf{A}_{m+\frac{1}{2}}^{n}(\mathbf{f}_{m+2}^{n}-3\mathbf{f}_{m+1}^{n}+3\mathbf{f}_{m}^{n}-\mathbf{f}_{m-1}^{n}) - \frac{t}{2}\mathbf{B}_{m+\frac{1}{2}}^{n} \tag{22}$$

and

$$\mathbf{f}_{m}^{n+1} = (1-\Omega_1)\mathbf{f}_{m}^{n} + \frac{\Omega_1}{2}(\mathbf{f}_{m+\frac{1}{2}}^{n+\frac{1}{2}}-\mathbf{f}_{m-\frac{1}{2}}^{n+\frac{1}{2}}) - \frac{t}{h}\mathbf{A}_{m}^{n+\frac{1}{2}}(\mathbf{f}_{m+\frac{1}{2}}^{n+\frac{1}{2}}-\mathbf{f}_{m-\frac{1}{2}}^{n+\frac{1}{2}}) - t\mathbf{B}_{m}^{n+\frac{1}{2}}. \tag{23}$$

In (22) and (23), Ω and Ω_1 are disposable constants such that when $\Omega = \Omega_1 = 0$, (22) and (23) will reduce to the original equations of Lax–Wendroff type. The provisional values of $\mathbf{A}$ and $\mathbf{B}$ can be evaluated by averaging values at data points such that

$$\mathbf{A}^n_{m+\frac{1}{2}} = \tfrac{1}{2}(\mathbf{A}^n_{m+1} + \mathbf{A}^n_m), \quad \mathbf{A}^{n+\frac{1}{2}}_m = \tfrac{1}{2}(\mathbf{A}^{n+\frac{1}{2}}_{m+\frac{1}{2}} + \mathbf{A}^{n+\frac{1}{2}}_{m-\frac{1}{2}}),$$

and similarly for $\mathbf{B}$.

It is known that the convergence of explicit finite-difference discretizations of hyperbolic equations will be ensured if the Courant–Friedrichs–Lewy (CFL) condition is satisfied by choosing a step size in the marching direction such that

$$|\lambda_{\max}|\,xt/h \leqslant 1.$$

It can further be shown that this condition is coincident with the stability requirements of the original difference equations of Lax–Wendroff type, e.g. by using the von Neumann stability analysis. However, the CFL condition for stability is necessary but not sufficient.

4. Results and discussion

The finite-difference equations (22) and (23) were solved by a Fortran program on a UNIVAC 1106 computer. The details of the program are given elsewhere (Biringen 1975). The method was tested by comparing the program predictions with the data of Biringen (1975) for an axisymmetric jet in a coflowing air stream and the data of Chevray (1968) for the turbulent wake of a body of revolution. In the former case measured distributions of U, k and $\overline{uv}$ at an axial distance of ten nozzle diameters were used as the initial conditions. The distributions of L were inferred from the measurements of the one-dimensional energy spectrum of the u-component of the fluctuating velocity. Chevray's (1968) measurements of U, k and $\overline{uv}$ at an axial station three diameters downstream of the body were chosen as the initial conditions for the wake calculations. The distributions of the length scale L in this case were obtained from the dissipation, which in turn was inferred from the production of turbulent kinetic energy.

It should be noted that in § 2 values were assigned to all the constants except the diffusion constants C_2 and C_2'. In addition to this, the diffusion functions $f_1(y)$ and $f_2(y)$ have to be prescribed as empirical inputs. For the jet calculations, these inputs were optimized for $m = 0{\cdot}1$, where m is the ratio of the velocity of the outer stream to the jet exit velocity. In all the subsequent calculations pertaining to jets, identical empirical inputs were used.

Numerical experiments were performed to optimize the empirical inputs by setting C_2 and C_2' equal to one and modifying f_1 and f_2, having made an initial estimate of these functions from the data of Wygnanski & Fiedler (1969) for the axisymmetric free jet. The functions f_1 and f_2 were prescribed in terms of $\eta = y/\delta$, where δ is the half-thickness, and were calibrated at every step in the integration procedure. The following conditions had to be met.

(i) In order to obtain increasing kinetic energy levels on the centre-line, $f_1(\eta)$ had to be allowed to attain negative values. Although this is not a direct outcome of the correlation measurements of Wygnanski & Fiedler (1969), it is compatible with the idea that energy should be transported away from regions of maximum energy (Bradshaw 1975, private communication).

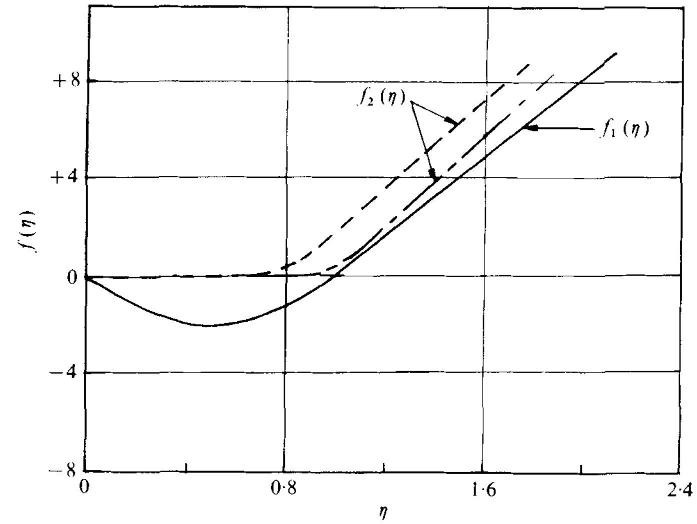


FIGURE 2. Diffusion functions. - · -, jets; - -, wakes.

(ii) The shear-stress profiles were found to be insensitive to $f_2(\eta)$. However, because near the edges of the flow $\lambda_4 = (V + C_2' V_J)/U$ is the outermost characteristic, V_J had to be adjusted to allow for the rate of spread of the flow.

(iii) The diffusion term in the shear-stress rate equation is zero on the centre-line. This term is given by

$$\frac{\partial(\overline{uv^2})}{\partial y} = C_2' \frac{\partial(\overline{uv}V_J)}{\partial y} = C_2' \left[\overline{uv} \frac{\partial V_J}{\partial y} + V_J \frac{\partial \overline{uv}}{\partial y} \right].$$

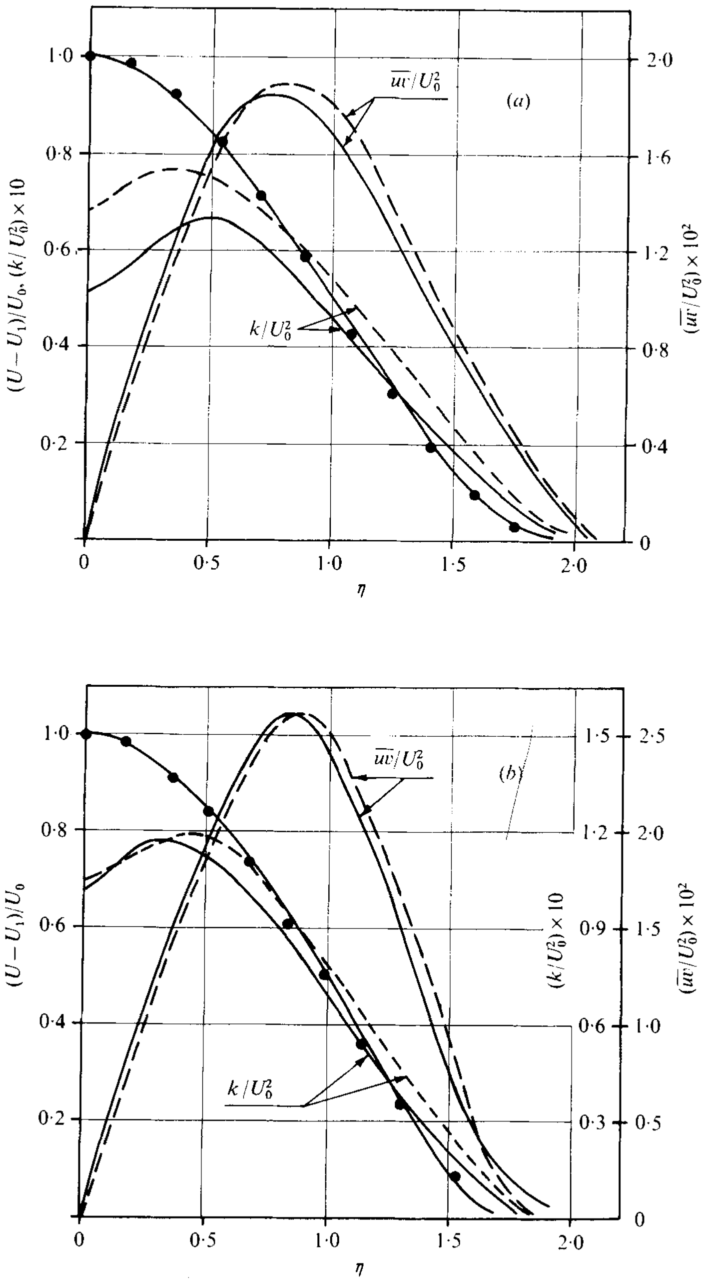
But on the centre-line $\overline{uv} = 0$ and $\partial \overline{uv}/\partial y \neq 0$, hence zero diffusion requires

$$V_J|_{y=0} = 0, \quad f_2(\eta)|_{n=0} = 0.$$

(iv) The only modification to the empirical inputs for the wake calculations was the shift in the value of η at which $f_2(\eta)$ started to assume values other than zero (figure 2). This was adjusted to obtain the best fit to the measured mean velocity profiles. Adequate values were around $\eta = 0{\cdot}6$ for the wake calculations and $\eta = 1{\cdot}0$ for the jet calculations.

According to these considerations, the functions shown in figure 2 were found to be adequate to describe the diffusion of the kinetic energy and the shear stress. The profiles of mean velocity and shear stress were almost invariant with the values of these functions, however the profiles of kinetic energy were extremely sensitive, especially to the minimum of $f_1(\eta)$.

Calculations were at first performed by using the original Lax–Wendroff difference equations by setting $\Omega = \Omega_1 = 0$. The CFL condition was satisfied in choosing the step size in the x direction. However, for the jet calculations this scheme produced instabilities in the $\overline{uv}$ and k profiles around the axis of symmetry for large values of x/d (where d is the initial jet diameter), whereas a value of $\Omega_1 \simeq 0{\cdot}1$ was found to be adequate to restore stability. It should be noted that this is much less than $\Omega_1 = 1{\cdot}0$, which is the value used by Nash (1972). Figures 3(a), (b) and (c) show profiles of the



Figures 3(*a*, *b*). For legend see following page.

excess mean velocity, the shear stress and the kinetic energy for the axisymmetric jet at typical stations for $m = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}3$ respectively. The computed profiles are compared with the experimental results of Biringen (1975) and display close agreement for the mean excess velocity and the shear stress. The computed kinetic energy profiles, however, indicate a tendency for the kinetic energy to diffuse towards the centre-line at large x/d.

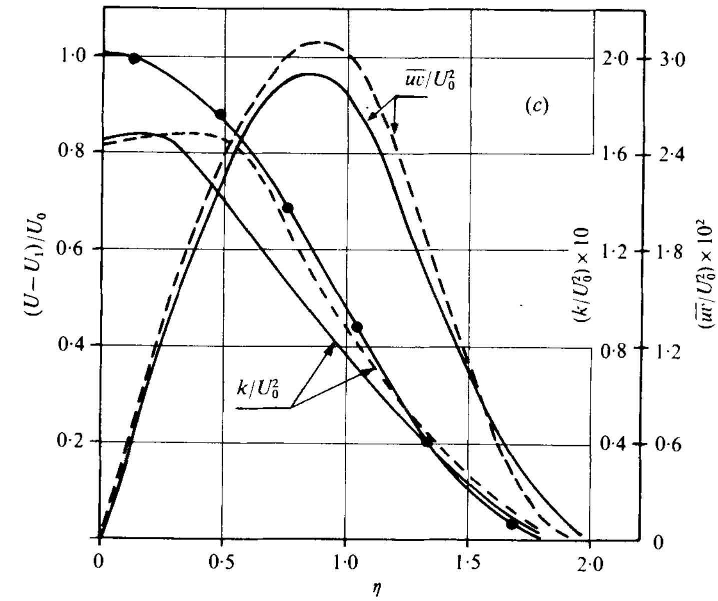


FIGURE 3. Axisymmetric jet: profiles of U, k and $\overline{uv}$. ——, calculations; - -, ●, measurements of Biringen (1975). (*a*) $m = 0{\cdot}1$, $x/d = 50$. (*b*) $m = 0{\cdot}2$, $x/d = 115$. (*c*) $m = 0{\cdot}3$, $x/d = 182$.

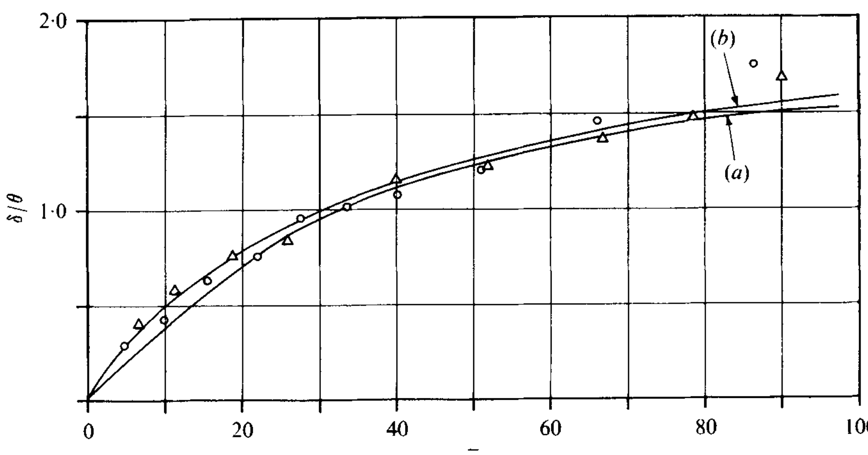


FIGURE 4. Axisymmetric jet: rate of spread. (*a*) Present calculations. (*b*) Predictions of Rodi (1972). ○, measurements of Biringen (1975); △, measurements of Antonia & Bilger (1973).

In most work on fully developed free turbulent flows it is assumed that at some distance downstream of the origin the flow development will depend only on the variable $\bar{x} = (x-x_0)/\theta$, where θ is the excess momentum thickness and for axisymmetric flows reads

$$\theta = \left[\frac{M}{\rho U_1^2}\right]^{\frac{1}{2}}, \quad M = 2\pi\rho\int_0^\infty U(U-U_1)\,y\,dy.$$

The initial conditions at the origin are assumed to affect only x_0, which is the location of the virtual origin.

In figures 4–7 the jet flow development is investigated as a function of $\bar{x}$. The com-

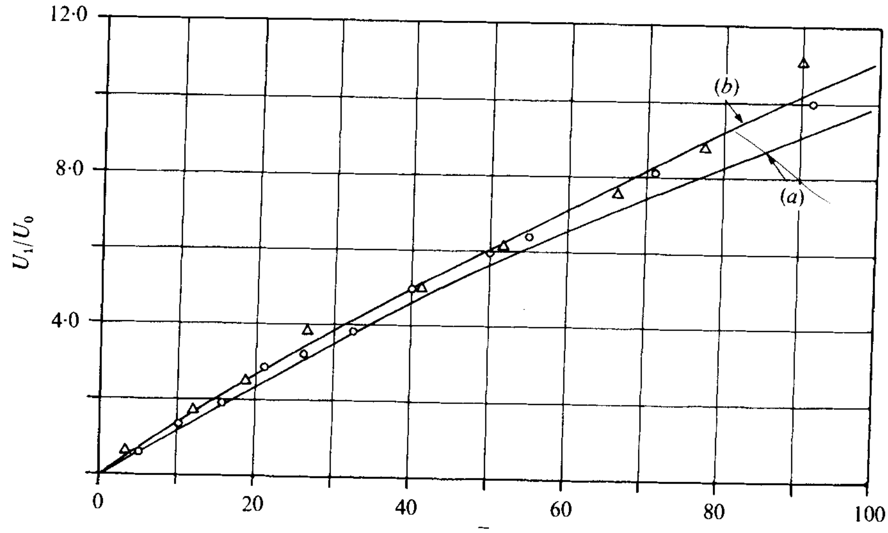


FIGURE 5. Axisymmetric jet: centre-line velocity decay. (*a*) Present calculations. (*b*) Predictions of Rodi (1972). ○, measurements of Biringen (1975); △, measurements of Antonia & Bilger (1973).

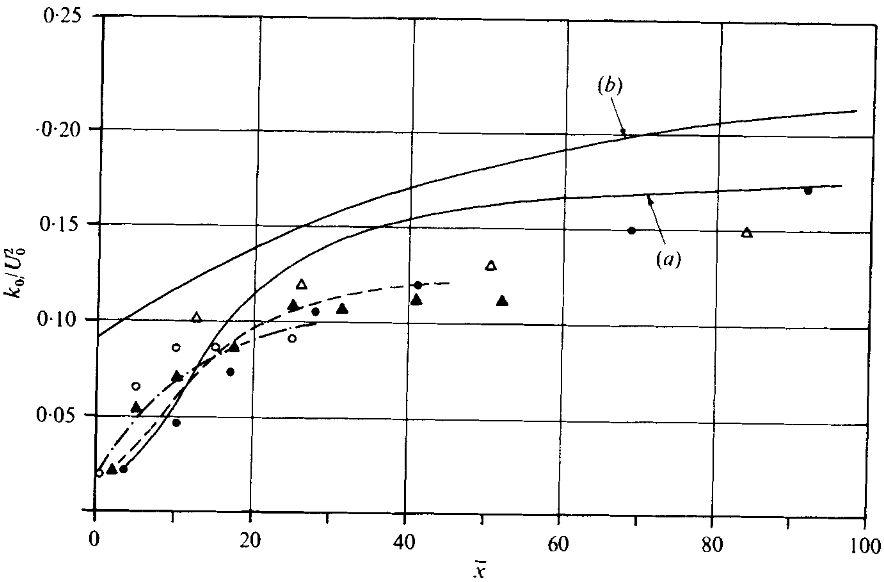


FIGURE 6. Axisymmetric jet: kinetic energy on the centre-line. -·-, present calculations for $m = 0{\cdot}1$; --, present calculations for $m = 0{\cdot}2$. (*a*) Present calculations for $m = 0{\cdot}3$. (*b*) Predictions of Rodi (1973). ○, ▲, ●, measurements of Biringen (1975) for $m = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}3$ respectively; △, measurements of Antonia & Bilger (1973).

puted profiles are compared with the predictions of Rodi (1972), which were obtained from a two-equation model of turbulence where the auxiliary variables are k and kL and $\overline{uv}$ is calculated from the Prandtl–Kolmogorov formula

$$\overline{uv} = -C_\mu k^{\frac{1}{2}} L \, \partial U/\partial y,$$

where C_μ is a constant to be determined.

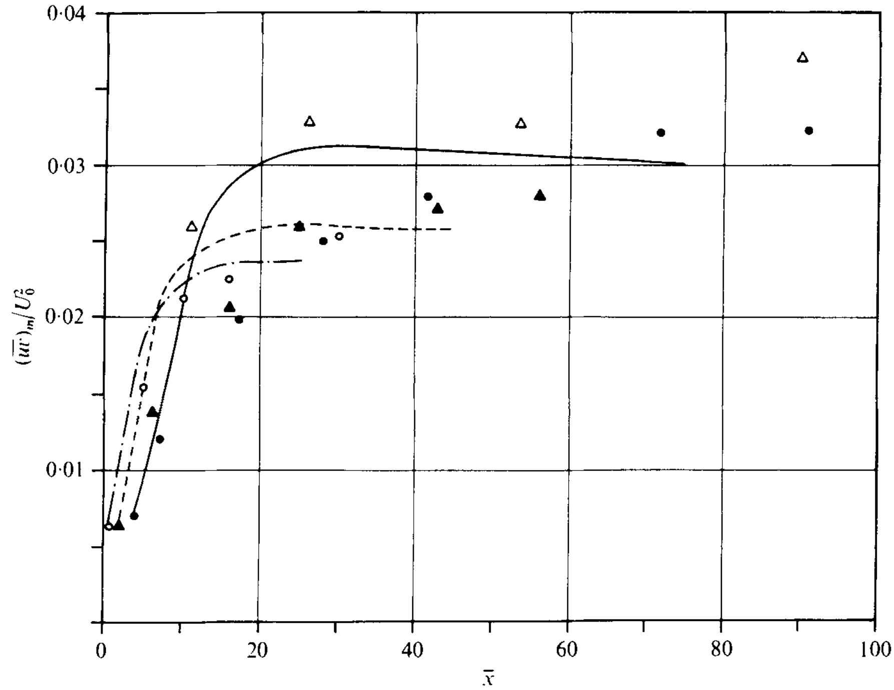


FIGURE 7. Axisymmetric jet: maximum shear stress. ---, present calculations for $m = 0{\cdot}1$; – – –, present calculations for $m = 0{\cdot}2$; —— present calculations for $m = 0{\cdot}3$; ○, ▲, ●, measurements of Biringen (1975) for $m = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}3$ respectively; △, measurements of Antonia & Bilger (1973).

It should be noted that the curves corresponding to the present calculations were obtained by superposing the results for different m. That the results for the variation of δ/θ and U_1/U_0, respectively, fall on a single curve indicates that the predictions are in accord with the expected universal behaviour of the mean flow parameters with $\bar{x}$. This is supported by the experiments of Biringen (1975) and Antonia & Bilger (1973) except at large $\bar{x}$, where the experimental errors are likely to be more pronounced.

The variations of the centre-line value k_0/U_0^2 of the turbulent kinetic energy and the maximum value $(\overline{uv})_m/U_0^2$ of the shear stress with $\bar{x}$ are shown in figures 6 and 7 respectively. For these quantities the results of the present calculations for different m follow the non-universal behaviour displayed by the experimental data, as well as predicting closely the limiting values of k_0/U_0^2. The same trends are observed in the variation of $(\overline{uv})_m/U_0^2$ with $\bar{x}$, but in this case for $m = 0{\cdot}3$ the calculated increase in the shear-stress level is more abrupt than that indicated by experiment.

Profiles of the mean velocity deficit, the shear stress and the kinetic energy for the axisymmetric wake are shown in figure 8. The predictions are compared with the experimental results of Chevray (1968) at $x/D = 15{\cdot}0$, where D is the maximum diameter of the body of revolution. Agreement of the predicted mean velocity with experiment is close, but the distributions of shear stress and kinetic energy display a larger departure from experiment than was observed in the set calculations; however the overall agreement of the predictions with experiment is satisfactory. In figures 9 and 10 a comparison of the present wake calculations is made with those for Pope &

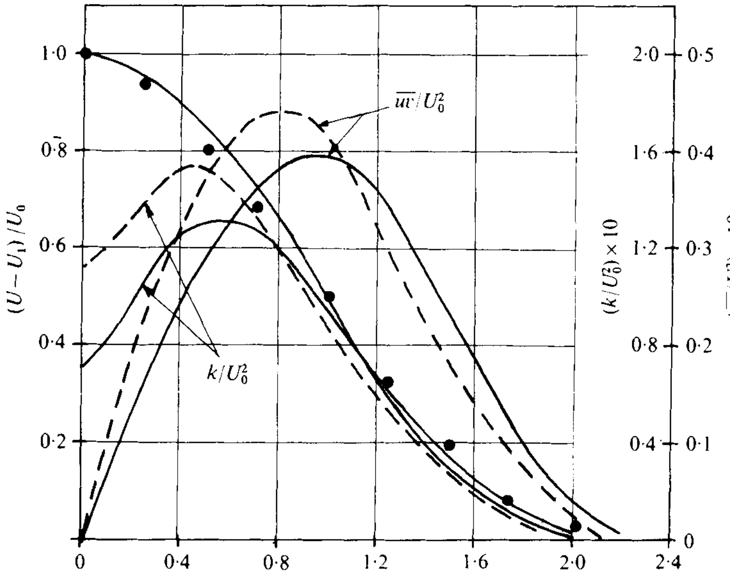


FIGURE 8. Axisymmetric wake: profiles of U, k and $\overline{uv}$ for $x/D = 15{\cdot}0$. ——, calculations; --, ●, measurements of Chevray (1968).

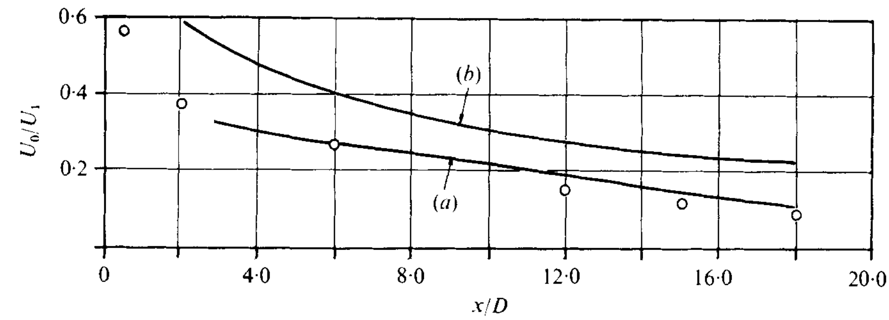


FIGURE 9. Axisymmetric wake: centre-line velocity decay. ○, measurements of Chevray (1968). (*a*) Present calculations. (*b*) Calculations of Pope & Whitelaw (1976).

Whitelaw's (1976) model III. Computational results are compared with the data of Chevray (1968). Both the streamwise development of the mean flow in terms of the decay of the centre-line velocity and the streamwise development of the turbulent structure in terms of the maximum shear stress are better predicted by the present method. For the wake calculations, the profiles of mean velocity and shear stress retained their self-similarity whereas the profiles of kinetic energy, k/U_0^2 *vs.* η, remained a function of the streamwise co-ordinate, mainly because of the rapid increase in k_0/U_0^2 at large x/D.

The primary difficulty encountered during the computations was due to the anomalous behaviour of the k profiles around the centre-line, which was more pronounced in the wake calculations. This could partly be attributed to the fact that the well-posed character of the differential equations is altered in a narrow region close to the symmetry axis. It should be noted that on the centre-line boundary conditions had to be prescribed for all the dependent variables U, k, $\overline{uv}$ and L (V is calculated

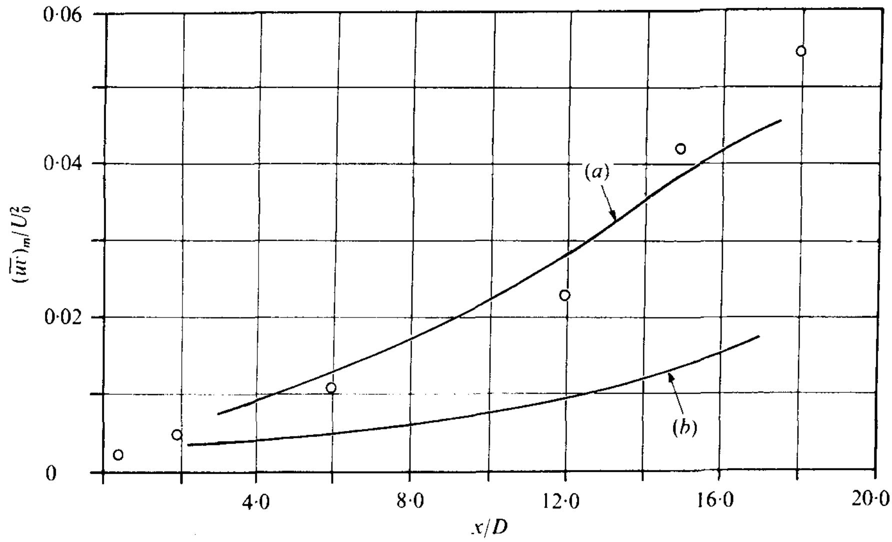


Figure 10. Axisymmetric wake: maximum shear stress. ○, measurements of Chevray (1968). (a) Present calculations. (b) Calculations of Pope & Whitelaw (1976).

separately from the equation along the vertical characteristic). However, Kreiss (1973) discusses the fact that for first-order hyperbolic systems well-posed problems should probably have the number of boundary conditions specified equal to the number of ingoing characteristics. In the present problem, when $y \to 0$ the characteristics given by (17) reduce to

$$\lambda_{1,2} = 0, \quad \lambda_{3,4} = \pm (ak)^{\frac{1}{2}}/U,$$

hence the double characteristic coincides with the symmetry line and the two others are inclined to the symmetry line at equal and opposite angles, so that the maximum number of boundary conditions allowed is three. The results, however, show that this affects the solution only in a very narrow region around the centre-line and only in the k profiles.

5. Concluding remarks

The main features of this work, which concerns the prediction of axisymmetric wakes and coflowing jets, are now summarized.

(i) The method is capable of predicting closely the mean flow field as well as the distributions and maximum values of the shear stress.

(ii) The centre-line values of the kinetic energy are also closely predicted but the k profiles display a tendency for the kinetic energy to diffuse towards the centre-line at large streamwise distances. This is partly attributed to the necessary alteration of the well-posed character of the differential equations in a narrow region around the symmetry axis.

(iii) Kinetic energy profiles are found to be very sensitive to the empirical input $f_1(\eta)$. Profiles of the mean velocity and $\overline{uv}$, however, are much less sensitive to the empirical inputs.

(iv) For this system of equations the Lax–Wendroff explicit scheme had to be modified to restore stability, lending support to previous work in the same direction (Nash 1972).

(v) In the case of a jet issuing into a still ambient fluid, since the characteristic $\lambda_3 \to \infty$ as $U \to 0$ special numerical treatment at the outer edge is necessary to march the solution in the streamwise direction.

The author wishes to thank Prof. J. Ginoux, Prof. J. J. Smolderen and Prof. D. Olivari for helpful discussions during the course of this work. Clarifying comments by Referees 1 and 2 are also acknowledged.

REFERENCES

ANTONIA, R. A. & BILGER, R. W. 1973 *J. Fluid Mech.* **61**, 805.

BIRINGEN, S. 1975 D.ScA. dissertation, Université Libre de Bruxelles.

BRADSHAW, P. 1973 *Imp. Coll. Aero Rep.* no. 73–05.

BRADSHAW, P., DEAN, R. C. & MCELIGOTT, D. M. 1972 *A.S.M.E. Paper* no. 72–WA/FE-25.

BRADSHAW, P., FERRISS, D. H. & ATWELL, N. P. 1967 *J. Fluid Mech.* **29**, 593.

CHAMPAGNE, F. H., HARRIS, V. G. & CORRSIN, S. 1970 *J. Fluid Mech.* **41**, 81.

CHEVRAY, R. 1968 *Trans. A.S.M.E., J. Basic Engng* **90**, 275.

CHOU, P. Y. 1945 *Quart. Appl. Math.* **3**, 38.

DALY, D. J. & HARLOW, F. H. 1970 *Phys. Fluids* **13**, 2634.

DONALDSON, C. P. 1971 *AGARD Conf. Proc.* no. 93, paper B-1.

HANJALIĆ, K. & LAUNDER, B. E. 1972 *J. Fluid Mech.* **52**, 609.

KREISS, H. O. 1973 *AGARD Lecture Ser.* no. 64, paper 1.

LAUNDER, B. E., REECE, G. J. & RODI, W. 1975 *J. Fluid Mech.* **68**, 537.

LUMLEY, J. & KHAJEH-NOURI, B. 1974 *Adv. Geophys.* **184**, 169.

MOREL, T. & TORDA, T. P. 1974 *A.I.A.A. J.* **12**, 533.

NASH, J. F. 1969 *J. Fluid Mech.* **37**, 625.

NASH, J. F. 1972 *Trans. A.S.M.E., J. Basic Engng* **94**, 131.

POPE, S. B. & WHITELAW, J. H. 1976 *J. Fluid Mech.* **73**, 9.

RICHTMYER, R. D. & MORTON, K. W. 1967 *Difference Methods for Initial Value Problems.* Wiley.

RODI, W. 1972 Ph.D. dissertation, University of London.

ROTTA, J. C. 1951 *Z. Phys.* **129**, 547.

ROTTA, J. C. 1971 *AGARD Conf. Proc.* no. 93, paper A-3.

ROTTA, J. C. 1975 Lecture presented at von Kármán Inst., Belgium, no. VKI LS-76.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow*, 1st edn. Cambridge University Press.

UBEROI, M. S. 1957 *J. Appl. Phys.* **28**, 1165.

WYGNANSKI, I. & FIEDLER, H. E. 1969 *J. Fluid Mech.* **38**, 577.